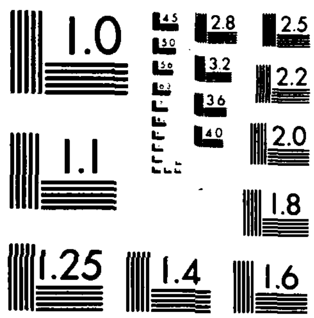
AD-A138 951 A NOTE ON THE 'CORELESSNESS' OR ANTIBALANCE OF GAME(U) 1/1
YALE UNIV NEW HAVEN CT COWLES FOUNDATION FOR RESEARCH
IN ECONOMICS M SHUBIK ET AL. 27 SEP 83
UNCLASSIFIED DISCUSSION PAPER-678 N00014-77-C-0518 F/G 12/1 NL

END
DATE
FILMED
4 84
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD A138951

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 678
Contract N00014-77-C-0518

A NOTE ON THE "CORELESSNESS" OR ANTIBALANCE OF GAME

Martin Shubik and Shlomo Weber

September 27, 1983

discusses aspects. END

~~A NOTE ON~~ THE "CORELESSNESS" OR ANTIBALANCE OF GAME*

by

Martin Shubik and Shlomo Weber

1. MARKET GAMES AND SIMPLE GAMES

The characteristic function of a market game has the special property that the game is totally balanced. Every one of the 2^n-1 nonempty subgames which can be formed with the n players of an n-person market game has a core. No matter what groups are considered, there is always some set of imputations at which all gain and no other group can do better for its members. The core leaves room for the bargain where all subgroups can have their "we can go it alone" claims satisfied. When an economy with an efficient price system is modeled as a game, the resultant game is totally balanced. There appears to be an intimate relationship between the design of an economic mechanism that can be efficiently run by prices and totally balanced games. This document

to the n power

The price system and the vote appear to be the two characterizing features of a private ownership democratic society. Yet the key class of games which characterize voting is considerably different from market

---

*This work relates to Department of the Navy Contract N00014-77-C-0518 issued by the Office of Naval Research under Contract Authority NR 047-006. However, the content does not necessarily reflect the position or the policy of the Department of the Navy or the Government, and no official endorsement should be inferred.

games. A simple game has two kinds of coalitions, winning and losing. Its characteristic function can be described by

$$v(S) = \begin{cases} 0 & \text{if } S \text{ is losing} \\ 1 & \text{if } S \text{ is winning.} \end{cases}$$

A simple game satisfying the property that the complement of any winning set is losing is called proper. If it satisfies the condition that the complement of any losing set is winning, it is called strong. A strong proper simple game is decisive, there will be no deadlocks or ties.

A key feature of simple games is that unless there are veto players present, they do not have cores. This contrasts with market games where cores are omnipresent.

The price system in a private ownership economy is designed to allow for efficient decentralized trade, where implicit in the model is the acceptance of the initial ownership claims as legitimate.

Simple games in contrast with market games characterize communal decisions. Voting in a democratic society is used to make joint or communal decisions about public goods and services and the societal needs and wants as a whole. In fact, the stark win, lose aspect of simple games is generally not encountered in much of voting. Rules exist to protect minorities. The strongest of these rules appears to be the veto.

The unanimity game is one where every single individual is fully protected by being able to exercise a veto. Although on one occasion the Polish nobility employed the veto in the election of their king, in general, democratic governments and even committees avoid the unanimity rule because too many stalemates can be created and in actuality the

speed of decisionmaking may be of importance.

A different way other than the veto for the protection of the minority is by limitations of the powers of the majority over the minority. Thus, the "winners, keepers" aspect of the simple game is modified to limit the winning coalition's ability to exploit the losers. An important example of such minority protection is in corporate law where although the control of a corporation goes to a majority when profits are paid out, they must be disbursed in proportion to shares held.

In the remainder of this note, both for mathematical simplicity and ease of exposition, most of our remarks and computations are confined to symmetric games with or without sidepayments. We use $v(S)$ or alternatively $f(s)$ to indicate the sidepayment characteristic function. As all coalitions of the same size yield the same payoff, the number $s = |S|$ can be used instead of the set S .

Figure 1 shows five basic symmetric games, whose characteristic functions are noted below:

$$\text{(1a)} \quad f(s) = \begin{cases} 0 & \text{if } s \le n/2 \\ 1 & \text{if } s > n/2 \end{cases} \qquad \text{(1b)} \quad f(s) = \begin{cases} 0 & \text{if } s < n \\ 1 & \text{if } s = n \end{cases}$$

$$\text{(1c)} \quad f(s) = \begin{cases} 0 & \text{if } s \le n/2 \\ s/n & \text{if } s > n/2 \end{cases} \qquad \text{(1d)} \quad f(s) = s/n \quad \forall s$$

$$\text{(1e)} \quad f(s) = \begin{cases} 1 & \text{if } s > \frac{n}{2} \\ \frac{1}{2} & \text{if } \frac{n}{2} \ge s > \frac{n}{4} \\ \vdots & \\ \frac{1}{2^m} & \text{if } \frac{n}{2^m} \ge s > \frac{n}{2^{m-1}} \end{cases}$$

where m is the only natural number such that $2 \leq n/2^m < 1$. Two of these games (1a) and (1b) are simple games, (1a) is decisive for n odd, (1b) is not. Three of these games (1b), (1c) and (1d) are market games. Game (1d) is inessential.

Games (1a) and (1e) are, respectively, the simple majority voting game and the game we call the generalized majority game v^0. Both of these games have no core. Game (1b) is the unanimity game and every point in the set of imputations is in the core. Game (1c) is closely related to the stockholder protection law. A majority is required for control, but they can only reward themselves in proportion to the size of their majority. This yields a one point core as can be seen from $f(s)/s \leq f(n)/n$ for all s.

We suggest that putting the core back into a simple game is a way to reconcile political and economic considerations. In the simple example illustrated in Figure (1c) as a modification of (1a), the rule is precisely correct for not merely turning the system for votes. But this, of course, is what is called for, for shares in jointly but privately held corporations. Essentially Arrow and Debreu (1954), by distributing profits in proportion to shares held converted the voting game into an inessential game as shown in (1d) where control voting has no significance.

We may consider a class of games related to simple games, but with the modification that although all losing coalitions still can obtain nothing by themselves the winning coalitions cannot necessarily take all. Thus, a better description of them is as control coalitions rather than winning coalitions. Thus, a *control game* is decisive if the complements of all control coalitions are losing, or can guarantee

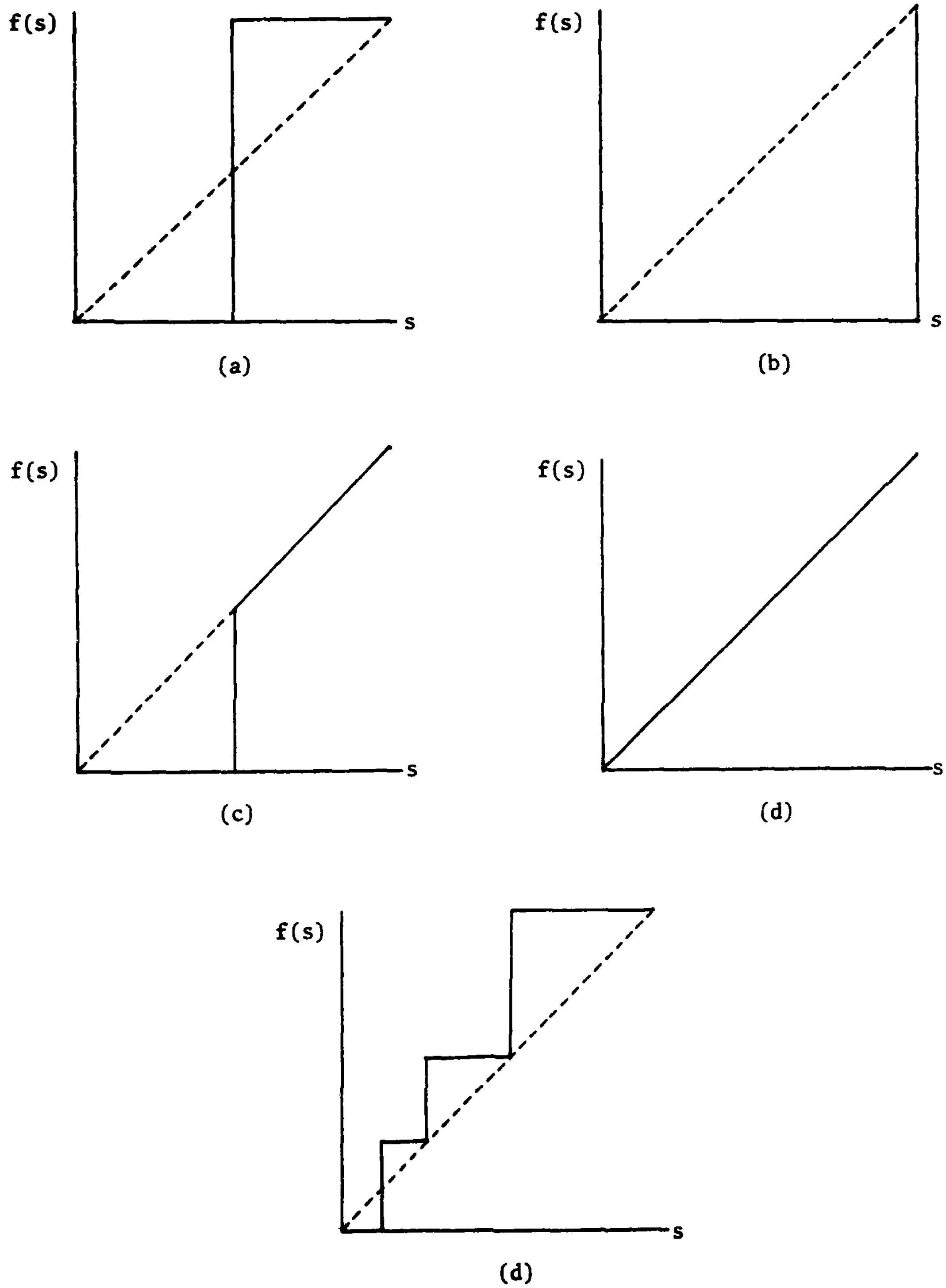


FIGURE 1

themselves zero, while the complements of all losing coalitions are control coalitions. Thus, the game in Figure (1c) is a decisive control game and a market game; while the unanimity game (1b) is a control game and a market game, but not decisive.

In the remainder of this note, we raise two sets of questions; one relatively broad, the other narrow. We limit ourselves here to offering answers only to the second.

The first set of questions concern the relationship between the price mechanism and the vote. We know that at least for sidepayment games that for any market game, we can find an exchange economy which gives rise to it, and vice versa. This does <u>not</u> tell us that if we start with a model involving voting production and exchange, and show that it is a market game that we will be able to find a price for votes. The prices may be for a set of artificial commodities with little direct relation to the original votes and endowments. Yet when we observe that it is possible by introducing stockholder protection to obtain prices for shares there seems to be a hope that by putting in the core in an appropriate way modifying majority rule with minority protection, we can obtain market games. Under what circumstances can we select minority protection rules such that a price for votes can be found?

If there is no minority protection, it is highly unlikely that a price for votes can emerge, as was shown by Shubik and Van der Heyden (1978). Logrolling offers an exchange possibility for votes, but not enough to produce prices.

Our second set of questions which are narrower and explicitly technical are to some extent preliminary to the questions raised above.

We have noted that market games form an extremely special subset

of all games. The requirement for total balance is extremely strong. In some sense simple games without veto players appear to be at the other end of the spectrum. In particular, the simple majority voting game appears to be about as "coreless" or as "unbalanced" as we can find. We make this description precise below and construct, for the symmetric case the game with the fewest balanced subgames.

It is worth noting that total unbalance is not feasible. All one person games have cores, and for a superadditive characteristic function all two person games have cores. The first meaningful instance of a simple majority game is for $n = 3$. In economics (and in mating) two appears to be a very special number. Markets and marriages appear to reflect the special importance of a two-sided relationship, but although in several democracies two party systems appear to be important, in politics and voting in general the two-sided relationship does not seem to be an important characteristic.

2. MEASURES OF DIVERGENCE FROM TOTAL BALANCE

2.1. Numbers of Coalitions

Restricting ourselves to the symmetric game, we can consider two measures of "distance" from a market game by (1) counting the number of coalitions of different size without cores, where we count all coalitions of any specific size only once. Thus, in total, we examine only n coalitions for an n-person game. Alternatively, we could count all 2^n-1 coalitions; in which case surpressing the cores of coalitions of size around $n/2$ becomes important.

Consider $\mathcal{A}_n$ --a class of all symmetric superadditive n-person games with sidepayments. For any $v \in \mathcal{A}_n$, denote

$A(v) = \{1 \leq k \leq n \mid \text{core of any } k\text{-person subgame of } v \text{ is empty}\}$. We claim that the maximal cardinality of $A(v)$ over $\mathcal{A}_n$ is $n - [\log_2 n] - 1$.

<u>Theorem 1</u>. $\max_{v \in \mathcal{A}_n} |A(v)| = n - [\log_2 n] - 1$. Moreover, the maximum is achieved at v^0 , the generalized majority game.

<u>Proof of Theorem 1</u>. Let $v \in \mathcal{A}_n$ be given. For any $1 \leq k \leq n$ denote by $\bar{v}_k = v_k/k$, where $v_k = v(s)$ for any k person coalition S .

Choose k_1 such that $\bar{v}_k \leq \bar{v}_{k_1}$ for any $k < k_1$ and no $k > k_1$ with $\bar{v}_k \geq \bar{v}_{k_1}$. Clearly, $k_1 > n/2$, otherwise, by superadditivity of v , $\bar{v}_{2k_1} \geq \bar{v}_{k_1}$, a contradiction to the choice of k_1 .

Choose now $k_2 < k_1$, such that $\bar{v}_k \leq \bar{v}_{k_2}$ for any $k < k_2$, and no $k_2 < k < k_1$ with $\bar{v}_k \geq \bar{v}_{k_2}$. Clearly $k_2 \geq k_1/2$, otherwise, superadditivity implies $\bar{v}_{2k_2} \geq \bar{v}_{k_2}$, $2k_2 \leq k_1$, again a contradiction to the choice of k_2 .

In the same way we choose $k_3, \ldots, k_{r-1}, k_r$. (Note that $k_{r-1} = 2$, $k_r = 1$ and the sequence $\{k_\ell\}$, $\ell = 1, 2, \ldots, r$ is uniquely defined by the game v .) We claim that:

(i) all k_1-, k_2-, ..., k_r-person supergames have nonempty cores;

(ii) if k does not coincide with any k_ℓ , $1 \leq \ell \leq r$, the core of any k-person subgame is empty.

<u>Proof of (i)</u>. Let k_ℓ $(1 \leq \ell \leq r)$ be given and consider any k_ℓ-person subgame (without loss of generality $\{1, 2, \ldots, k_\ell\}$). Let B be a balanced collection of the k-person subgame with this set of players.

$$\sum_{S \in B} \delta_S v(S) = \sum_{i=1}^{k_\ell} \sum_{\substack{S \in B \\ i \in S}} \delta_S \cdot \frac{v(S)}{|S|} ,$$

where δ_S is the balancing coefficient of S .

According to the choice of k_ℓ , $v(S)/|S| \leq \bar{v}_{k_\ell}$ for any S with no more than k_ℓ players. Using this remark and the fact that $\sum_{\substack{S \in B \\ i \in S}} \delta_S = 1$ for any i , we have

$$\sum_{i=1}^{k_\ell} \sum_{\substack{S \in B \\ i \in S}} \delta_S \frac{v(S)}{S} \leq \sum_{i=1}^{k_\ell} \bar{v}_{k_\ell} = k_\ell \bar{v}_k = v_{k_\ell} .$$

Since the last inequality holds for any balanced collection B , all k_ℓ-person subgames are balanced. Then the Shapley-Bondareva theorem ensures the nonemptiness of the core for all k_ℓ-person subgames.

Q.E.D.

Proof of (ii). Consider an arbitrary k , which does not coincide with any k_ℓ , $\ell = 1, 2, \ldots, r$, and any k-person subgame, without loss of generality, $\{1, 2, \ldots, k\}$. k does not belong to $A(v)$, therefore there exists $p < k$ with $\bar{v}_p > \bar{v}_k$. Consider a set of all p-person subsets of $\{1, 2, \ldots, k\}$. This collection is trivially balanced and

$$\sum_{|S|=p} \delta_S v_p = \sum_{i=1}^{k} \sum_{\substack{|S|=p \\ S \ni i}} \delta_S \frac{v_p}{p} = \sum_{i=1}^{k} \bar{v}_p > k_k = v_k .$$

We conclude that this k-person subgame is nonbalanced, and by Shapley-Bondareva theorem, the core of it is empty.

Q.E.D.

By (i) and (ii), $A(v) = N \setminus \{k_1, \ldots, k_r\}$. Let M be such that $2^M \leq n < 2^{M+1}$. ($M = [\log_2 n]$.) Since $k_1 > n/2$, or $k_1 \geq n+1/2$ and $k_\ell \geq k_{\ell-1}/2$ for $\ell = 2, \ldots, r$, it follows that $r \geq M+1$. It

is easy to see that if $v = v^0$, the generalized majority game $r = M+1$ is achieved. Hence

$$\max_{v \in A_n} |A(v)| = |A(v^0)| = n - M - 1 = n - [\log_2 n] - 1 .$$

Q.E.D.

Now let us turn to a question of a maximal number of subgames with the empty core over A_n. For any $v \in A_n$ denote by $c(v)$ a number of all subgames of v with the empty core. Using notations of the proof of Theorem 1, $c(v) = 2^n - 1 - \sum_{\ell=1}^{r} \binom{n}{k_\ell}$. We are abl o give the asymptotical estimation of $\max c(v)$. Formally,

Theorem 2.

$$\max_{v \in A_n} c(v) = \begin{cases} 2^n - \frac{3}{\sqrt{\pi n}}\lambda^n + o(\lambda^n), & \text{if } n = 3m, \\ 2^n - \frac{3}{2^{1/3}\sqrt{\pi n}}\lambda^n + o(\lambda^n), & \text{if } n = 3m+1, \\ 2^n - \frac{3}{2^{2/3}\sqrt{\pi n}}\lambda^n + o(\lambda^n), & \text{if } n = 3m+2, \end{cases}$$

where $\lambda = 3^{1/3}(3/2)^{2/3}$, and m is a natural number.

Let us note that if $\max c(v)$ is achieved then $n \geq k_1 > n/2$, $k_\ell = [(k_{\ell-1}+1)/2]$, $\ell = 2, \ldots, r$ and we consider only such sequences of $k_1, k_2, \ldots, k_r$. In order to prove Theorem 2 we need two following claims:

Claim 1. $\sum_{\ell=3}^{r} \binom{n}{k_\ell} = o\left[\binom{n}{k_2}\right]$, when $n \to \infty$.

Proof of Claim 1. Consider the ratio

$$R = \binom{n}{k_2} \Big/ \binom{n}{k_3} = \frac{(n - k_3) \times \dots \times (n - k_2 + 1)}{(k_3 + 1) \times \dots \times k_2} .$$

Since $k_3 < n/4$, $x \leq \frac{5}{4}k_3$ implies that $\frac{n-x}{x+1} \geq 2$ if n is large enough. Hence $R \geq 2^{\left[\frac{5}{4}k_3\right]}$. Note that $\sum_{\ell=3}^{n} \binom{n}{k_\ell} \leq k_3 \binom{n}{k_3}$ and therefore,

$$\sum_{\ell=3}^{n} \binom{n}{k_\ell} \Big/ \binom{n}{k_2} \leq \frac{k_3}{R} \leq \frac{k_3}{2^{\left[\frac{5}{4}k_3\right]}}$$

and this expression converges to 0 when k_3 tends to ∞ .

Q.E.D.

Claim 2. There exists n_0 such that $n > n_0$ implies

$$\operatorname*{Min}_{k>\frac{n}{2}, k_2 \geq \frac{k_1}{2}} \left\{ \binom{n}{k_1} + \binom{n}{k_2} \right\} = \binom{n}{K} + \binom{n}{2K} , \quad \text{where} \quad K = \left[\frac{n+2}{3}\right] .$$

In other words, minimum is achieved, when $k_2 = K$, $k_1 = 2K$.

Proof of Claim 2. We shall show that if $\tilde{k}_2 \neq K$, the above minimum is not achieved at $k_2 = \tilde{k}_2$.

1) Let $\tilde{k}_2 < K$. Then $\tilde{k}_1 = 2\tilde{k}_2 > n/2$. Denote $k_2' = n - 2\tilde{k}_2$, $k_1' = 2n - 4\tilde{k}_2$. It is easy to check that $k_1' > n/2$. We have $\binom{n}{k_2'} + \binom{n}{k_1'} = \binom{n}{2\tilde{k}_2} + \binom{n}{k_1'} = \binom{n}{\tilde{k}_1} + \binom{n}{4\tilde{k}_2 - n}$. Since $4\tilde{k}_2 - n < \tilde{k}_2$, $\binom{n}{k_2'} + \binom{n}{k_1'} \quad \binom{n}{\tilde{k}_1} + \binom{n}{\tilde{k}_2}$ and minimum is not achieved if $\tilde{k}_2 < K$.

2) Now consider $\tilde{K}_2 = K+m$, where $m \geq 2$. Then

$$\binom{n}{\tilde{K}_2} \Big/ \binom{n}{K} = \frac{K!(n-K)!}{\tilde{K}_2!(n-\tilde{K}_2)!} \geq \frac{(n-K-1)(n-K)}{(K+1)(K+2)}$$ and since $K < n/3 + 1$,

for n large enough the last ratio exceeds 2. Therefore,

$$\binom{n}{K} + \binom{n}{2K} \leq 2\binom{n}{K} < \binom{n}{\tilde{K}_2} < \binom{n}{\tilde{K}_2} + \binom{n}{2\tilde{K}_2}.$$

3) The last case $k_2 = K+1$. We will prove that for n large enough

$\binom{n}{K} + \binom{n}{2K} < \binom{n}{K+1} + \binom{n}{2K+2}$. It is easy enough to check that

$$\binom{n}{K+1} - \binom{n}{K} = \frac{n-2K-1}{K+1}\binom{n}{K}, \quad \binom{n}{2K} - \binom{n}{2K+2} = \frac{(4+2)(4K+1-1)}{(K+1)(4K+2)}\binom{n}{2K}.$$

Since $\binom{n}{K} \geq \binom{n}{2K}$, $\lim\limits_{n\to\infty} \frac{n-1-2K}{K+1} = 1$ and $\lim\limits_{n\to\infty} \frac{(n+2)(4K+1-n)}{(K+1)(4K+2)} = \frac{3}{4}$,

we concluded.

Q.E.D.

<u>Proof of Theorem 2</u>. *It follows from Claims 1 and 2, that*

$$\max_{v\in A_n} c(v) = 2^n - \binom{n}{K} - \binom{n}{2K} + o\binom{n}{K}.$$

There are three possible representations of n : 1) $n = 3m$; 2) $n = 3m+1$; 3) $n = 3m+2$, where m is a natural number. Consider all the cases.

<u>Case 1</u>. $\binom{n}{K} = \binom{n}{2K}$ and $\binom{n}{K} + \binom{n}{2K} = 2\binom{n}{n/3} = 2\frac{n!}{\left(\frac{n}{3}\right)!\left(\frac{2n}{3}\right)!}$. By Stirlings formula the last expression is equivalent to

$$\frac{2\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi\frac{n}{3}}\sqrt{2\pi\frac{2n}{3}}\left(\frac{n}{3e}\right)^{n/3}\left(\frac{2n}{3e}\right)^{2n/3}} = \frac{3}{\sqrt{\pi n}}\lambda^n \quad \text{where } \lambda = 3^{1/3}\left(\frac{3}{2}\right)^{2/3}.$$

Case 2. $\binom{n}{2K} = o\binom{n}{K}$, where $n \to \infty$.

$$\binom{n}{K} = \binom{n}{\frac{n}{3}+\frac{2}{3}} = \frac{n!}{\left(\frac{n}{3}+\frac{2}{3}\right)!\left(\frac{2n}{3}-\frac{2}{3}\right)!} \sim$$

$$\text{(Stirling's formula)} \sim \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi\left(\frac{n}{3}+\frac{2}{3}\right)}\left(\frac{n}{3}+\frac{2}{3}\right)^{\frac{n}{3}+\frac{2}{3}}\sqrt{2\pi\left(\frac{2n}{3}-\frac{2}{3}\right)}^{\frac{2n}{3}-\frac{2}{3}}} \sim$$

$$\sim \frac{3\cdot\lambda^n}{2\sqrt{\pi n}\left(\frac{n}{3}+\frac{2}{3}\right)^{\frac{2}{3}}\left(\frac{2n}{3}-\frac{2}{3}\right)^{-\frac{2}{3}}} \sim \frac{3\cdot\lambda^n}{2^{\frac{1}{3}}\sqrt{\pi n}} .$$

The same computations in the Case 3 provide us the desirable formula:

$$\binom{n}{K} \sim \frac{3}{\sqrt{\pi n}\ 2^{2/3}}\ n .$$

This completes the proof of Theorem 2.

Q.E.D.

2.2. ε-Core Measure of Distance from Total Balance

We may try for a direct measure of how much we have to add to all coalitions to turn the original game into the "closest" totally balanced game.

For this purpose we make use of the notion of the strong ε-core which has been introduced by Shapley and Shubik (1966). That is, *the strong ε-core of game* v is the set of all payoff vectors $x \in R^n$ satisfying $\sum_{i \in S} x_i \geq v(S) - \varepsilon$ for all $S \subsetneq N$.

Consider B_n , the class of all normalized symmetric superadditive n-person games. (A game v is *normalized* if $v(i) = 0$ for each

$i = 1, 2, \ldots, n$ and $v(N) = 1$.) Let $v \in B_n$ be given and let $v_k \in B_k$ be its k person subgame with $k \leq n$. We denote

$$\varepsilon_k(v) = \mathrm{Min}\{\varepsilon \geq 0 \mid \text{the strong } \varepsilon\text{-core of } v_k \text{ is nonempty}\} .$$

And now we introduce the measure of deviation from total balancedness:

$$\rho(v) = \sum_{i=1}^{n} \varepsilon_k(v) .$$

The main result of this section is that v^0 , the generalized majority game, is the most unbalanced one with respect to ρ , i.e.

<u>Theorem 3</u>. Let $n \geq 2$. Then for any $v \in B_n$, $v \neq v^0$ implies $\rho(v) < \rho(v^0)$.

<u>Proof</u>. Let $v \in B_n$ be given. The game v determines a unique decreasing sequence of natural numbers $K(v) = \{k_0, k_1, \ldots, k_r\}$ such that

(1) $k_0 = n+1$, $k_r = 1$ and $k_\ell \geq k_{\ell-1}/2$ for each $\ell = 1, 2, \ldots, r$.

(2) for all $\ell = 1, 2, \ldots, r$, $\bar{v}_k \leq \bar{v}_k$ if $k \leq k$ and $\bar{v}_k < \bar{v}_k$ if $k_\ell < k < k_{\ell-1}$.

(Note that if $v = v^0$, then $k_0^0 = n+1$, $k_\ell^0 = [n \smallsetminus 2^\ell] + 1$ for $\ell = 1, 2, \ldots, r$, where, by Theorem 1, $r = [\log_2 n] + 1$.)

As it was shown in the proof of Theorem 1, a k-person subgame v_k of v has a nonempty core if and only if $\bar{v}_k \geq \bar{v}_h$ for all $1 \leq h \leq k$. Therefore v_k has a nonempty core if and only if $k \in K(v)$. Thus,

$$\varepsilon_k(v) = k\bar{v}_{k_\ell} - v(k) = k(\bar{v}_{k_\ell} - \bar{v}_k) \quad \text{for all } \ell = 1, 2, \ldots, r \text{ and } k_\ell \leq k < k_{\ell-1} .$$

By monotonicity of v, which is implied by superadditivity, $v(k) \geq v(k_\ell)$ for $k \geq k_\ell$ and hence for all $\ell = 1, \ldots, r$ and $k_\ell \leq k < k_{\ell-1}$, $\varepsilon_k(v) \leq k\bar{v}_{k_\ell} - v(k_\ell) = \bar{v}_{k_\ell}(k - k_\ell)$. Since for any $v \in B_n$, $\bar{v}_k \leq 1/k$ for each $k = 1, 2, \ldots, n$ and $\bar{v}^0_k = 1/k$ for all $k \in K(v^0)$, the equality $\varepsilon_k(v^0) = \bar{v}^0_k(k - k_\ell)$ for all $k_\ell \in K(v^0)$ and $k_\ell \leq k < k_{\ell-1}$ imply the assertion of Theorem 3 for all $v \in B_n$ with $K(v) = K(v^0)$. Therefore in order to complete the proof of the theorem, it suffices to consider the case where $K(v)$ does not coincide with $K(v^0)$.

Suppose, in negation, that there is $v^1 \in B_n$, such that $K(v^1) \in K(v^0)$ and $\rho(v^1) > \rho(v^0)$. Let $\rho(v^1) = \underset{v \in B_n}{\text{Max}}\, \rho(v)$. (Such a maximum exists since $\rho(\cdot)$ is a continuous function over compact subset of R^n.) Let m be a minimal number such that $k^1_m \neq k^0_m$ (and hence $k^1_m > k^0_m$). There are two possible cases:

(i) $k^1_{m-1} - k^1_m \geq k^1_m - k^1_{m+1}$,

(ii) $k^1_{m-1} - k^1_m < k^1_m - k^1_{m+1}$.

<u>Case (i)</u>. Consider a game v^2 for which $v^2_k = v^1_k$ for $k < \bar{k} = k^1_{m-1}$ and $k \geq k^1_{m-1}$, $v^2_k = (\bar{v}^1_{\bar{k}+1} + \delta)\bar{k}$ for $\bar{k} \leq k < k^1_{m-1}$, where $\delta > 0$ is small enough so that $v^2_{\bar{k}} < v^1_{k^1_{m-1}}$. Clearly, $K(v^2) = \{k^2_\ell\}_{\ell=1,\ldots,r}$, where $k^2_\ell = k^1_\ell$ for $\ell \neq m$, and $k^2_m = \bar{k} = k^1_{m-1}$. $K(v^2)$ satisfies (1) and (2) since $k^2_m \geq k^0_m > k^2_{m-1}$.

Since $\rho(v^1)$ is maximal, $\varepsilon_k(v^1) = \bar{v}_{k^1_\ell}(k - k^1_\ell)$ for all $k^1_\ell \leq k < k^1_{\ell-1}$ and $\bar{v}_{k^1_\ell} = 1/k^1_\ell$, $\ell = 1, 2, \ldots, r$. As $\varepsilon_k(v^1) = \varepsilon_k(v^2)$

for all $k < \bar{k}$ and $k \geq k^1_{m-1}$, we have $\rho(v^2) - \rho(v^1) =$

$$= \sum_{k=\bar{k}}^{k^1_{m-1}-1} \{\varepsilon_k(v^2) - \varepsilon_k(v^1)\} = \sum_{k=\bar{k}-1}^{k^1_{m-1}-1} \{((\bar{k}+1)^{-1} + \delta)(k-\bar{k}) - (\bar{k}+1)^{-1}(k-\bar{k}-1)\}$$

$$- \bar{v}^1_{k^1_{m+1}} (\bar{k} - k^1_{m+1}) > (\bar{k}+1)^{-1}(k^1_{m-1} - 1 - k) - \bar{v}^1_{k^1_{m+1}} (\bar{k} - k^1_{m+1}) >$$

$> (\bar{k}+1)^{-1}(k^1_{m-1} - 2k^1_m + k^1_{m+1})$, which is, by (i), positive. Therefore, $\rho(v^2) > \rho(v^1)$, *a contradiction to the maximality of* v^1, which completes a consideration of Case (i).

<u>Case (ii)</u>. Assume that $k^1_m - k^1_{m+1} > \Delta = k^1_{m+1} - k^1_m$. Let $\tilde{k} = k^1_{m+1} + \Delta$ and define a game v^3 as follows:

$$v^3_k = v^1_k \quad \text{if} \quad k < \tilde{k} \quad \text{or} \quad k \geq k^1_{m-1}$$

$$v^3_k = (\bar{v}^1_{k^1_m} + \lambda) \cdot \tilde{k}\,, \quad \text{if} \quad \tilde{k} \leq k < k^1_{m-1} \quad \text{where } \lambda > 0 \text{ is small enough so that } v^3_{\tilde{k}} \leq v^1_{k^1_{m-1}}\,.$$

Note that $k^1_{m+1} > k^1_m/2$ implies that $\tilde{k} = k^1_{m+1} + \Delta > k^1_m/2 + \Delta/2 =$ $= k^1_m/2 + (k^1_{m-1} - k^1_m)/2 = k^1_{m-1}/2$. Hence $K(v^3) = \{k^3_\ell\}$, $\ell = 1, 2, \ldots, r$ where $k^3_\ell = k^1_\ell$ if $\ell \neq m$ and $k^3_m = \tilde{k}$. Clearly $\varepsilon_k(v^1) = \varepsilon_k(v^3)$ if $\tilde{k} \leq k$ or $k \geq k^1_{m-1}$. Therefore $\rho(v^3) - \rho(v^1) = \sum_{k=\tilde{k}}^{k^1_{m-1}-1} \{\varepsilon_k(v^3) - \varepsilon_k(v^1)\}$.
But it is easy to see that, since $\bar{v}^3_{\tilde{k}} > \bar{v}^1_{k^1_m}$, it follows that

$\varepsilon_{\tilde{k}+h}(v^3) > \varepsilon_{k^1_{m+h}}(v^1)$ for $1 \leq h \leq \Delta$ and $\varepsilon_{\tilde{k}+\Delta+h}(v^3) > \varepsilon_{\tilde{k}+h}(v^1)$ for $1 \leq h \leq k^1_{m-1} - \tilde{k}$. Therefore $\rho(v^3) > \rho(v^1)$, again a contradiction which completes the proof of the theorem.

Q.E.D.

REFERENCES

Arrow, K. J. and G. Debreu (1954). "Existence of an Equilibrium for a Competitive Economy," Econometrica, 22: 265-290.

Shapley, L. S. and M. Shubik (1966). "Quasi-cores in a Monetary Economy with Nonconvex Preferences," Econometrica, 34(4): 805-827.

Shubik, M. and L. Van der Heyden (1978). "Logrolling and Budget Allocation Games," International Journal of Game Theory, 7(3/4): 151-162.